Outgrowths of the DLMF Project: Part 2:
NIST Digital Repository of Mathematical Formulae

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Challenges in 21st Century:
Experimental Mathematical Computation

Institute for Computational and Experimental Research in Mathematics, Providence, Rhode Island

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Online compendium of mathematical formulae
- orthogonal polynomial and special function formulae

DRMF attempts to use Web 2.0 technologies to move beyond the static presentation of reference data to a platform that encourages community interaction and collaboration.

DRMF utilizes of DLMF LaTeX macros
- tie specific character sequences to well-defined mathematical objects.
- Provides an internet link to standard, precise orthogonal polynomial and special function definitions through the DLMF and DRMF

Uses MediaWiki wiki software
- MathML support
- LaTeXXML
- MathJax
The DRMF will be designed for a mathematically literate audience and should:

1. facilitate interaction among a community of mathematicians and scientists interested in formulae data related to orthogonal polynomials and special functions (OPSF);
2. be expandable, allowing the input of new formulae;
3. be accessible as a standalone resource;
4. have a user friendly, consistent, and hyperlinkable viewpoint and authoring perspective; and
5. contain easily searchable mathematics and take advantage of modern MathML tools for easy to read, scalably rendered mathematics.
Mathematical Optical Character Recognition project

- Bateman manuscript project: Higher Transcendental Functions, Tables of Integral Transforms
- Byrd & Friedman’s Handbook of Elliptic Integrals for Engineers and Scientists

DLMF \LaTeX{} macro replacement project

- Hypergeometric Orthogonal Polynomials and Their $q$-Analogues – KLS
- KLS addendum by Tom Koornwinder
- future?: Andrews, Askey & Roy : Special Functions
- future?: Ismail : Classical and Quantum Orthogonal Polynomials in One Variable
- future?: etc.

Wikitext generation project

- NIST Digital Library of Mathematical Functions (ch. 25) : 170 formulas
DLMF macros provide semantic content in formulas

- **DLMF OPSF Macros via \[\LaTeX XML\]-server**
  - 546 semantic DLMF \[\LaTeX\] OPSF macros
  - additional 38 semantic \[\LaTeX\] macros

- **Objects:** \[\sum, \int, \text{deriv}\{f\}\{x\}, \text{qderiv}\{n\}\{q\}@\{z\}\]

- **Constants:** \[\text{expe}, \text{iunit}, \text{cpi}, \text{EulerConstant}\]

- **Special Functions and Orthogonal Polynomials**
  
  \[
  \Gamma(z) \quad \text{EulerGamma}@\{z\} \quad \text{http://dlmf.nist.gov/5.30#E1}
  
  J_\nu(z) \quad \text{BesselJ}\{\nu\}@\{z\} \quad \text{http://dlmf.nist.gov/10.2#E2}
  
  Q_{\nu}^\mu(z) \quad \text{LegendreQ}[\mu]\{\nu\}@\{z\}: \quad \text{http://dlmf.nist.gov/14.3#E7}
  
  P_{n}^{(\alpha, \beta)}(x) \quad \text{JacobiP}\{\alpha\}\{\beta\}\{n\}@\{x\} \quad \text{http://dlmf.nist.gov/18.3#T1.t1.r3}
  
  \]
Whereas *Wikipedia* and other web authoring tools manifest notions or descriptions as first class objects, the DRMF does that with mathematical formulae.

**DRMF** provides for each formula, a formula home page:

1. **Rendered description of the formula** (required);
2. **Constraints** the formula must obey;
3. **Substitutions** required to understand formula;
4. **Bibliographic citation** (required);
5. Open section for **proofs** (required) – *DLMF*;
6. **List of symbols** and **links** to definitions (required) – *DLMF macros*;
7. Open section for **notes** – *connections between formulas*; and
8. Open section for **external links** – *computer generated proofs*;
(-1)^r \zeta(r) (1 - s) = \frac{2}{(2\pi)^r} \sum_{n=0}^{\infty} \left( \sum_{m=1}^{\infty} \frac{1}{m^r} \right) \Re \left( e^{i \pi m} \cot \left( \frac{1}{2} \pi x \right) + \Im \left( e^{i \pi m} \csc \left( \frac{1}{2} \pi x \right) \right) \right)

Substitution(s) [edit]

\text{c} = \ln (2\pi) - \frac{1}{2} \pi

Constraint(s) [edit]

s \neq 0, 1, \text{ and } k = 1, 2, 3, ...

Proof [edit]

We ask users to provide proof(s), reference(s) to proof(s), or further clarification on the proof(s) in this space.

Symbols List [edit]

(-1) : f(t) = \text{sign}

\zeta : \text{Riemann zeta function}

\pi : \text{the ratio of a circle's circumference to its diameter}

\binom{n}{k} : \text{binomial coefficient}

\Re : \text{real part}

\cos : \text{cosine function}

\Im : \text{imaginary part}

\sin : \text{sine function}

\Gamma : \text{Euler's gamma function}

\ln : \text{principal branch of logarithm function}

\text{i} : \text{imaginary unit}

Bibliography [edit]

Equation (5), Section 25.4 of DLMF.

URL links [edit]

We ask users to provide relevant URL links in this space.
Further questions

- How does one facilitate effective community interaction & contribution with such a resource?
  - implement a high degree of computer verification of community input
  - ensure a degree of moderation in the wiki

- Can one build a piece of intelligent software which is able to
  - scan in books;
  - produces \LaTeX\ source;
  - replaces commands for functions in the source with semantic macros;
  - extracts data from the text (such as constraints)
  - associates data with relevant formulae and removes text;
  - produces Wikitext;
  - and uploads Wikitext to a publicly accessible website?

- How does one search the resulting mathematical database?
Ongoing projects to investigate the above questions

- **Macro replacements** from well-constructed \LaTeX{} source
- **Extraction of mathematical data** from text (keywords)
- **Wikitext generation**
- **Porting the DLMF search engine** in MediaWiki (DRMF)
- **Output of formula data** from right-clickable menus in a variety of formats so that formulas can be used and also verified
  - \LaTeX{} expanded
  - \LaTeX{} semantic
  - presentation MathML
  - content MathML
  - Mathematica
  - Maple
  - Sage
Virtual Machine Instances:

- **XSEDE** project
  - 2 **XSEDE CentOS**: demo and deployment
  - 2 **XSEDE Ubuntu server**: \(\LaTeX\)XML, Mathoid

- **Wikimedia Foundation** – Wikitech
  - 4 **WMF Vagrant instances**
Acknowledgements

- Moritz Schubotz (TU-Berlin): MediaWiki Math
- Bruce Miller (NIST): DLMF Macros
- Janelle Williams (VSU): 2013 SURF student

High School Students:
- Jake Migdall – MathJax menu customization
- Cherry Zou – seeding/macro replacement
- Alex Danoff – seeding/macro replacement
- Amber Liu – MathJax menu customization
- Jimmy Li – mathematical search
Poster session with website demos: Wed. evening

NIST Digital Repository of Mathematical Formulae (DRMF)
Howard S. Cohl*, Marjorie A. McClain*, Bonita V. Saunders*, Moritz Schubotz‡, Janelle C. Williams†
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- A digital compendium of math formulae for orthogonal polynomials and special functions and associated math data. Uses Web 2.0 technologies to move beyond the static presentation of reference data to a platform that encourages community interaction and collaboration.
- Use of DLMF semantic $\LaTeX$ macro set for special functions and orthogonal polynomials e.g., \EulerGamma@{z} http://dlmf.nist.gov/5.2#E1 $p_{\alpha,\beta}^{\mu}(z)$ \jacobip{alpha}{beta}{n}@{x} http://dlmf.nist.gov/18.3#T1.t1.r3
- DRMF treats formulae as first class objects, describing them in formula home pages which contain:
  - Rendered description of the formula (required);
  - Open section for notes (required);
  - Open section for proofs (required);
  - List of symbols and links to definitions (required);
  - Wikipedia generation and semantic DLMF $\LaTeX$ macro replacement effort using IDL & Python.
- DLMF $\LaTeX$ macros already implemented for the DLMF Zeta chapter;
- Hypergeometric Orthogonal Polynomials and their $q$-Analogues - KLS addendum

Zeta and Related Functions Page

http://gw32.iu.xsede.org/index.php/Main_Page

http://www.siam.org/meetings/opsfa13
Plenary Speakers:
- Perry Delft, Courant Institute of Mathematical Sciences, New York University, USA
- Charles F. Dunkl, University of Virginia, USA
- Olga Holtz, Technische Universität Berlin, Germany
- Mouaad E.H. Iseal, University of Central Florida, USA
- Teresa E. Perez Fernandez, Universidad de Granada, Spain
- Sarah Post, University of Hawaii at Manoa, USA
- Nico Temme, Centrum Wiskunde & Informatica (CWI), The Netherlands
- Craig A. Tracy, University of California Davis, USA
- Lauren Williams, University of California Berkeley, USA
- Daniel W. Lozier, National Institute of Standards and Technology, USA
- Diego Dominici, State University of New York at New Paltz, USA
- Sarah Post, University of Hawaii at Manoa, USA

Organizing Committee:
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- Audrey Terras, University of California San Diego, USA
- Walter Van Assche, Katholieke Universiteit Leuven, Belgium
- Luc Vinet, University of Montreal, Canada

Themes: Orthogonal Polynomials and Special Functions, including aspects within:
- classical analysis
- approximation theory
- continued fractions
- potential theory
- spherical functions
- asymptotics
- Riemann-Hilbert problems
- random matrix theory
- integrable systems
- Painlevé equations
- orthogonal polynomials and special functions of several variables
- orthogonal polynomials associated with root systems
- spherical functions
- orthogonality on the complex plane
- multiple orthogonal polynomials
- Sobolev orthogonal polynomials
- stochastic processes
Outgrowths of the Digital Library of Mathematical Functions Project
Part 1

DLMF Standard Reference Tables
http://dlmftables.uantwerpen.be

Daniel Lozier
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Outline

• Introduction
  – DLMF
  – DLMF Standard Reference Tables
  – Algorithms and software

• Examples
  – Table generation
  – Table comparison

• Acknowledgements
NIST Digital Library of Math Fcns and
NIST Handbook of Math Fcns compared to
Abramowitz and Stegun, 1964
Welcome to the Prototype of DLMF Standard Reference Tables!

The DLMF Standard Reference Tables service aims to provide tables of numerical values for special functions, with guaranteed accuracy to high precision and computed over parameters and arguments of your choice.

4. Elementary Functions
7. Error Functions, Dawson’s and Fresnel Integrals
8. Incomplete Gamma and Related Functions
10. Bessel Functions
DLMF Std. Reference Tables

Accurate function values on demand

• Keyed to http://dlmf.nist.gov

• Web input of
  – Desired arguments
  – Desired precision

• Web output of
  – Enclosures computed to 5 digits higher precision or one of 5 rounding modes at desired precision
DLMF Std. Reference Tables

Accuracy tests of function values from other sources

• Web input of
  – Arguments and function values from an external source

• Web output of
  – Digit-by-digit comparison
  – Erroneous digits shown in red
  – Relative error
Algorithms and Software

- Special functions computed to arbitrary precision and guaranteed accuracy
- Real variables, decimal or binary arithmetic
- Algorithms depend on detailed error analysis with series and continued fractions
- For details see
Example

table generation

Bessel function $J_n(x)$, integer order, real argument
### Table Request

Choose a function: Bessel function $J_v$

$J_v(x)$ denotes the Bessel function of the first kind of order $v$; see DLMF (10.2.2). Computations are currently restricted to integer and half-integer orders $v$, and real values of $x$.

**Evaluate $J_v(x)$ as a Tabulation or Comparison?**

Get function arguments from form or data file?

Specify grid for function parameters and arguments:

- **Choose $v$:**
  - from list: 0 5 10 15 20 30 40 50 100

- **Choose $x$:**
  - from list: 1

Show output to **25** digits, using interval mode.
Computed values of $J_\nu(x)$

Computed using MpIeee, for $\nu$ being each of 0, 5, 10, 15, 20, 30, 40, 50, 100 and $x$ being 1.0; output to 25 digits, using interval mode; computation and output in base 10.

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$x$</th>
<th>$J_\nu(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0</td>
<td>7.65197 68655 79665 51449 7175 26103±08×10^{-1}</td>
</tr>
<tr>
<td>5</td>
<td>1.0</td>
<td>2.49757 73021 12344 31375 0655 40988±03×10^{-4}</td>
</tr>
<tr>
<td>10</td>
<td>1.0</td>
<td>2.63061 51236 87453 20699 7853 68779±03×10^{-10}</td>
</tr>
<tr>
<td>15</td>
<td>1.0</td>
<td>2.29753 15322 10344 44380 6556 20688±03×10^{-17}</td>
</tr>
<tr>
<td>20</td>
<td>1.0</td>
<td>3.87350 30085 24657 71891 4787 52995±04×10^{-25}</td>
</tr>
<tr>
<td>30</td>
<td>1.0</td>
<td>3.48286 97942 51482 90224 9676 48686±04×10^{-42}</td>
</tr>
<tr>
<td>40</td>
<td>1.0</td>
<td>1.10791 58511 28632 66217 5020 74447±02×10^{-60}</td>
</tr>
<tr>
<td>50</td>
<td>1.0</td>
<td>2.90600 49481 73239 39446 9381 57792±03×10^{-80}</td>
</tr>
<tr>
<td>100</td>
<td>1.0</td>
<td>8.43182 87896 26708 54923 5063 65845±09×10^{-189}</td>
</tr>
</tbody>
</table>
Example

table comparison

Matlab’s error function $\text{erf}(x)$, real argument
### Table Request

**Choose a function**  
error function

\[ \text{erf}\ x \text{ denotes the error function; see DLMF (7.2.1).} \]
Computations are currently restricted to real values of \( x \).

**Evaluate erf \( x \) as a**  
- Tabulation
- Comparison

Specify the data file to compare to:

[ Browse... ] Matlab-erf-6args-6vals-input-file.txt

**Note:** The lines in the data file must contain values for \( x \) and \( \text{erf}\ x \) in that order, separated by commas or spaces.

Show output to the number of digits in file, using

- round to nearest (even)
## Computed values of $\text{erf} \, x$

Computed using Mpieee (on odd lines), compared to $x$, $\text{erf} \, x$ (on even lines), for $x$, $\text{erf} \, x$ from file Matlab-erf-6args-6vals-input-file.txt; output to 17 digits, using round to nearest (even); computation and output in base 10.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\text{erf} , x$</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.25000 00000 00000 0×10^{-2}</td>
<td>7.04319 77722 38707 8×10^{-2} 7.04319 77722 38707 4×10^{-2}</td>
<td>5.7 ×10^{-17}</td>
</tr>
<tr>
<td>1.25000 00000 00000 0×10^{-1}</td>
<td>1.40316 20480 13338 2×10^{-1} 1.40316 20480 13338 0×10^{-1}</td>
<td>1.5 ×10^{-16}</td>
</tr>
<tr>
<td>1.87500 00000 00000 0×10^{-1}</td>
<td>2.09117 67705 93758 5×10^{-1} 2.09117 67705 93758 4×10^{-1}</td>
<td>4.8 ×10^{-17}</td>
</tr>
<tr>
<td>2.50000 00000 00000 0×10^{-1}</td>
<td>2.76326 39016 82369 3×10^{-1} 2.76326 39016 82369 6×10^{-1}</td>
<td>1.1 ×10^{-16}</td>
</tr>
<tr>
<td>3.12500 00000 00000 0×10^{-1}</td>
<td>3.41468 63350 15950 1×10^{-1} 3.41468 63350 15950 1×10^{-1}</td>
<td>0.0</td>
</tr>
<tr>
<td>3.75000 00000 00000 0×10^{-1}</td>
<td>4.04116 90943 48223 0×10^{-1} 4.04116 90943 48223 1×10^{-1}</td>
<td>2.5 ×10^{-17}</td>
</tr>
</tbody>
</table>
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  – Marje McClain
  – Bonita Saunders

• Primary Antwerp Collaborators
  – Franky Backeljauw
  – Stefan Becuwe
  – Annie Cuyt
Feedback

help us provide an excellent data service!!

• Digit display
  – Do you like or dislike?
  – Do you prefer just relative errors?

• Extra digits and color
  – Do you like or dislike?

• Please send ideas for
  – Further development
  – Further application areas